Naming Game on small-world networks: the role of clustering structure

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Naming Game is a recently proposed model for describing how a multi-agent system can converge towards a consensus state in a self-organized way. In this paper, we investigate this model on the so-called homogeneous small-world networks and focus on the influence of the triangular topology on the dynamics. Of all the topological quantities, the clustering coefficient is found to play a significant role in the dynamics of the Naming Game. On the one hand, it affects the maximum memory of each agent; on the other hand, it inhibits the growing of clusters in which agents share a common word, *i.e.*, a larger clustering coefficient will cause a slower convergence of the system. We also find a quantitative relationship between clustering coefficient and the maximum memory.

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I. INTRODUCTION

The semiotic dynamics is a new emerging field concerning the origination and evolution of languages [1, 2]. This field studies how semiotic relations originate and how they spread in a group of individuals. One of the most interesting questions in this area is whether and how the group of agents can converge towards a final consensus state despite an initial dissension. The Naming Game, as a minimal model, was proposed recently to study such convergence process [3]. The original model of Naming Game was presented by Steels *et al.* in a well-known artificial intelligence experiment called Talking Head [4]. In the primary experiment, humanoid robots observed the ambient objects via their digital cameras, and assigned them different names. It has been certified by the experiment that, under certain rules, the system will evolve to a consensus state, characterized by a shared lexicon among robots, without any human intervention, *i.e.*, the system can converge towards a state of consensus in a self-organized way.

In this paper, we focus on a special model of Naming Game proposed by Baronchelli *et al.* [5]. This model characterizes a non-equilibrium dynamical process of a complex adaptive system and allows the system to converge towards a special attractor in a self-organized way, so that, it has gained special attentions from physicists. This model has been previously investigated on some typical networks, such as fully connected graphs [5, 6], regular lattices [7], small-world networks [8], and heterogenous networks [9]. These works uncovered many interesting properties of Naming Game. Recent studies on network science suggest that network-based dynamics depend significantly on some special topological properties [10, 11, 12]. In the context of Naming Game, it has been found that the topological features of underlying network, such as the degree distribution, the clustering coefficient, the modularity and so on, play important roles in the dynamics [9]. In this paper, we focus on the clustering structures of the network. We study the model mentioned above on homogeneous small-world networks and investigate the relationship between clustering coefficient and some dynamical quantities of Naming Game.

This paper is organized as follows. In section 2, we describe the model mentioned above. In section 3 we provide both the simulation and analytical results. And finally, in section 4, we sum up this article and discuss some interesting issues that may be relative to the further works.

II. MODEL

The model proposed by Baronchelli *et al.* [5] captures the essential properties of current dynamics: the system converges towards a consensus state in a self-organized way. To keep the paper self-contained, we review the model briefly. The model includes a set of agents occupying the nodes of a network. Two agents are permitted to negotiate if they are connected by a link. For simplicity, it is assumed that there is only one object in the environment.

At each time step, the present model evolves through the following rules:

(1)An agent is randomly selected from the network to play the role of "speaker", and from the speaker's neighbors, another agent is randomly selected to be a "hearer";

(2)The speaker picks up a word, if there is any, from its local vocabulary and sends it to the hearer; if the speaker's vocabulary is empty, the speaker will invent a new word to name the object, add the new word to its own vocabulary, and then send it to the hearer;

(3)After receiving the word from the speaker, the hearer searches its local vocabulary to check whether it has already been recorded previously. If this is true, the negotiation will be deemed as a successful one, and both hearer and speaker will then delete all other words in their vocabularies; otherwise, the hearer will add the new

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received word to its local vocabulary without deleting any other words.

This process continues until there is only one word left. It is obvious that at the beginning, the system contains a variety of different names, however, it has been certified that, after enough steps of local interactions, the system will eventually share an exclusive word.

Networks adopted here are the so-called homogeneous small-world networks(HoSW) proposed by Santos *et al.* [13]. Slightly different from the original process, we construct our networks in the following way. Starting from a ring of N nodes, each connected to its K nearest neighbors, we choose a node and the edge that connects it to its nearest neighbor in a clockwise sense. With probability p, we exchange this edge with another unchanged edge (Detailed rules can be seen in Ref. [14]); otherwise, we leave it in place. We repeat this exchange by moving clockwise around the ring until one lap is completed. Afterward, we consider the edges that connect nodes to their second-nearest neighbors. As before, we exchanged each of these edges with an unchanged edge with probability p. We continue this process, circling around the ring and proceeding outward to further neighbors after each lap, until each edge has been visited once. We forbid any self-edges and multi-edges during this reconstruction.

Since the degree of each node keeps unchanged during the reconstruction process [13, 14], every node in HoSW network is of degree K. In our simulations, we keep $K = 8$ fixed. Similar to the Wattts-Strogatz(WS) networks [13, 15], when the exchange probability p is close to 0, a HoSW network has a clustering coefficient close to a regular lattice's, and when p increase to the order of 10^{-1}, a dramatic decrease of the clustering coefficient can be observed. However, it is still worth while to mention that network topologies in different backgrounds may differ from each other, and the reason we adopt HoSW networks is that, as mentioned above, they are able to bring an effective quantitative change of clustering coefficient together with preserving the degree of each node.

III. NAMING GAME ON HOSW NETWORKS

In this section, based on the model mentioned above, we investigate how dynamical properties of Naming Game response to the change of the network topology. Former works [9] on Naming Game had revealed that clustering coefficient and node degree are two of the most important factors for the dynamics. In this paper, we perform Naming Game on HoSW networks to eliminate the influence of varying degrees, and lay our special emphasis on the clustering coefficient. The local clustering coefficient of node i is defined as $C_i = 2e_i/k_i(k_i - 1)$, where e_i is the number of edges among i's neighbors, and k_i is the degree of node i (in this paper, $k_i \equiv K, \forall i$), and the average clustering coefficient of the network is $C = \frac{1}{N}\sum_i C_i$, where N is the number of nodes, and i runs over all the nodes [15]. Figure 1 shows the average

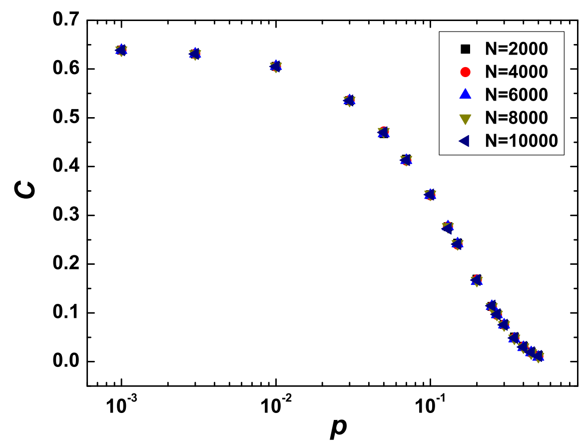

FIG. 1: (Color Online) Clustering coefficient as a function of rewiring probability p. For each N and p, the clustering coefficient is averaged over 20 independently generated networks.

clustering coefficient of networks with different sizes as a function of rewiring probability p. It indicates that the clustering coefficient is almost independent of the network size.

Before discussing the influence of topological quantities on the dynamical properties of Naming Game, a general view of the dynamics may be helpful. Two of the most relevant quantities for describing the dynamical process are the total number of words over the network $n(t)$, and the number of distinct words over the network $nd(t)$.

Figure 2 displays the evolution of these two quantities as a function of the time. In the early stage of the dynamics, a variety of words are invented by agents. Because many nodes have no words in this period, $n(t)$ keeps increasing and at a time scales as N, it reaches a maximum value. We denote this maximum by M, and the time $n(t)$ reaches it by t_{max}. After t_{max}, dominant word clusters begin to engulf puny ones, and at the same time aggrandize their sizes. This can be seen from the power-law decrease section of $nd(t)$ and the plateau region of $n(t)$ in Fig. 2. After this period of elimination, the system reaches a crossover point. We denote this cross time by t_{cross}. After the crossover time, the competition among the surviving words lasts until a final winner takes up the whole network and declares the convergence of the system. In this paper, we focus particularly on the period of word invention($t \sim t_{max}$) and the period of word elimination($t_{max} < t < t_{cross}$).

The first dynamical quantity we concerned is the maximum memory M, which measures the minimum storage ability each agent should have, or from another point of view, the minimum effort each agent should make in order to reach the consensus state. Previous works [7, 8] had found that the maximum memory strongly depends on the network topologies. In the following, we provide

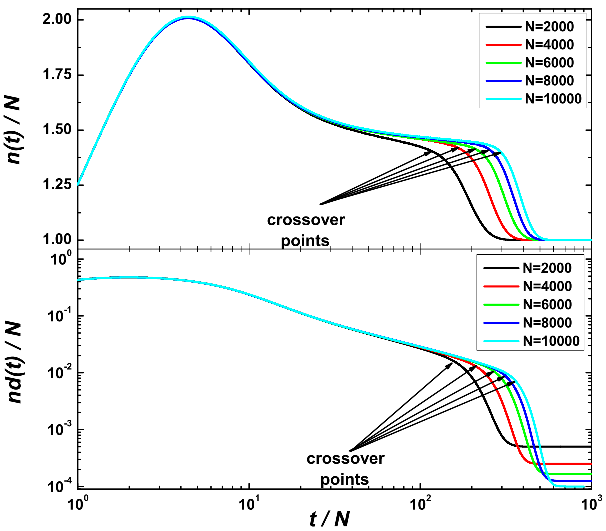

FIG. 2: (Color Online) The average words per agent $n(t)/N$ as a function of rescaled time t/N. $p = 0.1$, averaged over 5000 realizations realizations. The maximum value of $n(t)$ is proportional to N. (Color Online) The evolution of the rescaled number of different words $nd(t)/N$ for different network sizes a function of time. $p = 0.1$, averaged over 5000 independent realizations.

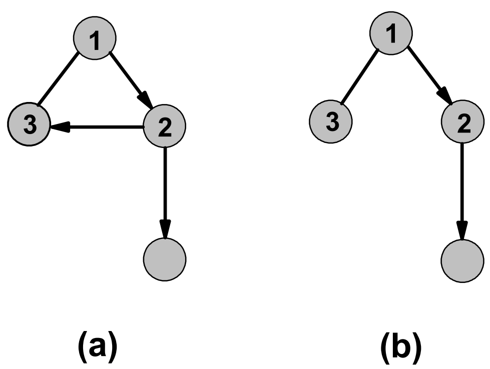

FIG. 3: The sketch map of the effect of clustering coefficients on the spreading process of words.

a quantitative relationship between clustering coefficient and the maximum memory on the basis of mean-field theory. Because the nodes' degrees of current networks are identical, the obtained relationship can avoid any influence of varying degrees on the concerned quantity.

Suppose that node i is selected to be the speaker at one time, we use ε_i to denote the vocabulary size of node i and η_{ij} to denote the size of the intersection set between the vocabularies of the speaker i and one of his neighbors j. If j is selected to be the hearer, the probability of a successful negotiation is η_{ij}/ε_i. Having considered that each neighbor of i has the same chance $1/k_i$ (here k_i is the degree of i) to be the hearer, we can formulate the successful probability at this time:

$$S_i = \frac{1}{k_i \varepsilon_i} \sum_{j \in \Omega_j} \eta_{ij} \tag{1}$$

where the sum runs over all the neighboring nodes of i (this set is indicated by Ω_j). At the early stage when words begin to spread from original nodes to the neighbors, the local success rate of node i depends strongly on its local clustering coefficient C_i. Suppose that a node has a highly connected neighborhood, the words it sent to its neighbors are more likely to be restricted in its neighborhood, thus contribute to the coordinate of its intersection sets. Figure 3 illustrates this presentation in a simple form: node 1 sends a word to its neighbor node 2, because hearers are randomly chosen from the speakers' neighborhood, in the situation of figure (a), node 2 has a chance to send this word to node 3, which is also a neighbor of node 1; while in the situation of figure (b), since the lack of triangular loops, the word sent to 2 can not be sent back to the neighborhood of 1. It is easy to generalize this simplest situation to more complex ones: the larger the clustering coefficient of a node, the larger the intersection set between its vocabulary and vocabularies of its neighbors. According to Eq. (1), larger intersection set, *i.e.*, higher values of η_{ij}, will result in higher local success rate.

Figure 4 shows the numerical results of the relationship between S_i and C_i at t_{max} in the network ensembles of different p: a linear positive correlation is obtained. We can simply present this relationship as $S_i = aC_i + b$ by averaging S_i according each local clustering coefficient C_i in the vast ensembles of networks and get $a \approx 0.35$, $b \approx 0.20$ (Note that: a and b are implicitly involving the factor of nodes' degree K and we fix $K = 8$ in the present work). Since speakers are randomly chosen from the network, the success rate of the step t_{max} can be written as $S(t_{max}) = \frac{1}{N}\sum_i S_i$, thus we can write $S(t_{max}) = 0.35C + 0.20$ in the network ensembles with different global clustering coefficients.

To get the relation between clustering coefficient and the maximum memory, we can first expect that the total number of words $n(t)$ and the success rate $S(t)$ satisfies the following rate equation:

$$\frac{dn(t)}{dt} = -2S(t)(\frac{n(t)}{N} - 1) + 1 - S(t), \tag{2}$$

where the first item in the right side denotes the reduction of total words with the success rate $S(t)$ and the second item denotes increasing one word with unsuccessful rate $1 - S(t)$. Here, we adopt the mean field approximation that each node has the same number of words $n(t)/N$

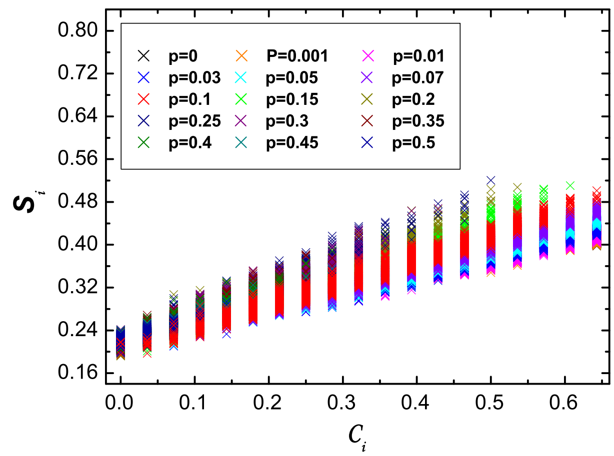

FIG. 4: (Color Online) Relationship between S_i and C_i at the specific time t_{max}. Networks size $N = 2000$. For each p, we generate 20 networks and on each network, we perform 5000 independent simulations. In a single simulation, once t reaches t_{max}, the local success rates of nodes are calculated according to Eq. (1), and S_i is obtained through averaging them over these simulations. Each scatter in the figure represents a node in the network sets. A linear positive correlation between S_i and C_i is suggested.

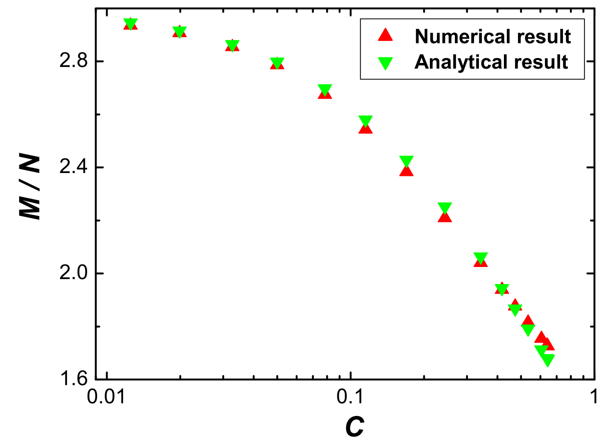

FIG. 5: (Color Online) Numerical(red) and analytical(green) results about maximum memory per agent M/N as a function of global clustering coefficient C. $N = 2000$, and p is regulated to special values to obtain suitable clustering coefficient. On each network, 5000 independent simulations are performed to obtain the averaged value of M.

(Note that: in the highly heterogeneous networks, the mean-field approximation may be invalid). Thus, each node will lose $n(t)/N - 1$ words and keep the successful one when the negotiation is successful as the dynamical rules. Furthermore, considering:

$$\frac{dn(t)}{dt}\,|_{t_{max}} = 0, \tag{3}$$

we obtain:

$$M = n(t_{max}) = \frac{N}{2}\left(1 + \frac{1}{S(t_{max})}\right). \tag{4}$$

Hence, the maximum memory can be obtained as:

$$M = \frac{N}{2}\left(1 + \frac{1}{aC + b}\right). \tag{5}$$

Figure 5 shows the average minimum memory per agent M/N, as a function of the global clustering coefficient C, and provides a consistent comparison between Eq. (5) and numerical results. Although the linear correlation between local success rate and local clustering coefficient is justified only in the situation of $N = 2000$, the relationship between M and C obtained by Eq.(5) can be generalized to networks with different sizes. Figure 6 shows M as a linear function of the network size N. According to the simulation results, we obtain $M = \alpha N$, with α being a constant in a fixed p. This indicates that $S(t_{max})$ is determined by rewiring probability p and independent of the network size N. Note that clustering coefficient

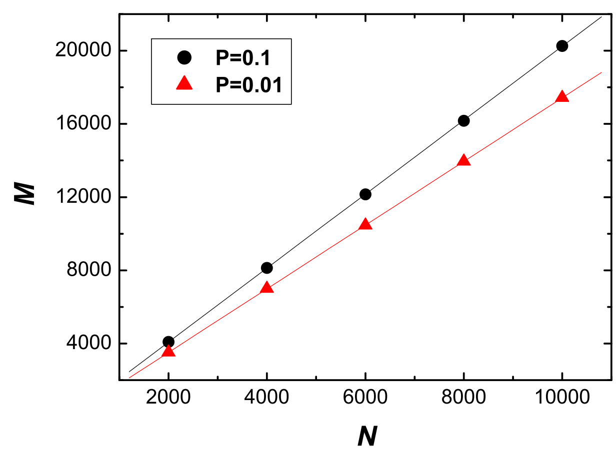

FIG. 6: (Color Online) Maximum memory M as a linear function of network size N. Each value of M is averaged over 5000 independent simulations. The functions of $p = 0.01$ and $p = 0.1$ are $M = 1.74N$ and $M = 2.02N$ respectively.

of HoSW networks is also independent of the network size N, as Fig. 1 has suggested. Thus we can generalize Eq. (5) to networks with any given sizes. By substituting the correspondent C of networks with $p = 0.01$ and $p = 0.1$ into Eq. (5), we obtain the correspondent values of $\alpha = 1.71$ and 2.06. They are very close to the simulation results 1.74 and 2.02 in Fig. 6.

The other dynamical quantity we concentrate on is the growing rate of word clusters. A word cluster is a set of neighboring agents sharing a word. Previous works had

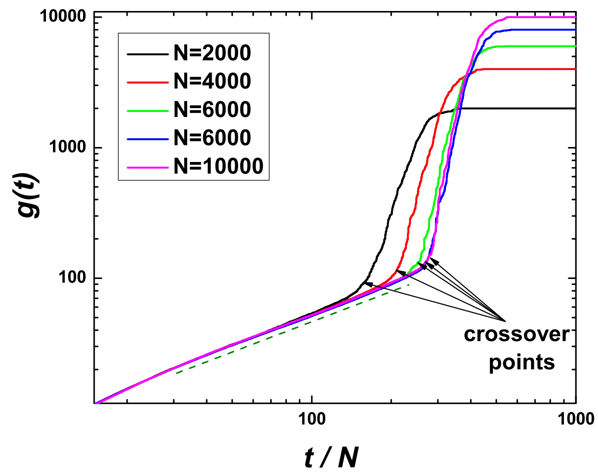

FIG. 7: (Color Online) Average cluster size $g(t)$ as a function of time. $p = 0.1$. Each curve is separated into two distinct sections by a crossover time. In the first section, a power-law increase of $g(t)$ can be observed.

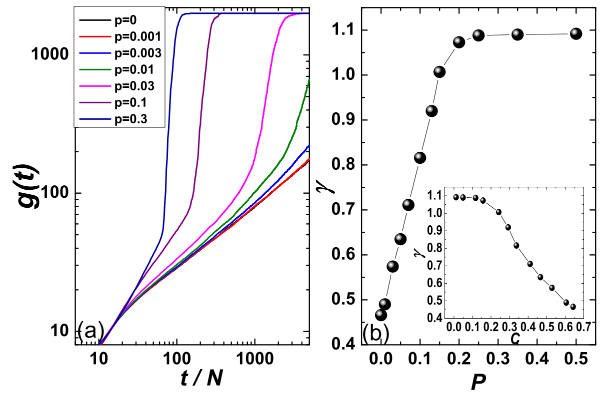

FIG. 8: (Color Online) Evolution of $g(t)$ as a function of time. Network size $N = 2000$. Growing rates obtained from networks with different rewiring probabilities differ from each other. A smaller p corresponds to a slower growing in the power-law section.

revealed that, if agents are embedded in low-dimensional lattices ($p \ll 1$), the convergence process of the system can be analogous to a coarsening of such clusters. The growing behavior of word clusters can be expressed as $g(t) \sim (t/N)^{\gamma}$, with $\gamma = 0.5$ in regular lattice [7, 8]. Here $g(t)$ is the average size of word clusters at time step t. Different from former works, in our situation, the dimension of network changes in a wide range, from regular lattices to random networks. Thus it is very interesting to investigate the behavior of word clusters in such networks, and the relationships between their growing behavior and network topologies. In the following, we also use $g(t)$, the average size of word clusters, to describe the evolution of dynamics. Figure 7 displays $g(t)$ as a function of time. The evolution of $g(t)$ is separated into two distinct sections by a crossover time t_{cross}. Having considered that dynamics after crossover time is highly affected by finite-size effects, we just pay attention to the first section of the evolution, where $g(t)$ displays a power-law growing behavior, as:

$$g(t) \sim (t/N)^{\gamma}, \quad (\tfrac{t}{N} < t_{cross}) \tag{6}$$

The index γ in equation (6) indicates the growing rate of word clusters. Figure 7 implies that when rewiring probability p reaches the order of 10^{-1}, the growth of word clusters still displays power-law behavior, as in the case of $p \ll 1$; however, the growing rate γ differs from the case of $p = 0$. Thus it is quite natural to expect γ depending on the network topology. Figure 8 shows that when rewiring probability p deviates from 0, word clusters will grow in a faster way. Note that in the model of HoSW networks, larger p yields smaller clustering coefficient, as Fig. 1 has suggested, we can expect a negative correlation between the growing rate and the clustering coefficient. The inset of Fig. 8 shows the relationship of these two quantities. It can be seen that a larger clustering coefficient corresponds to a slower growing of word clusters, and in the following paragraph, we try to give a qualitative explanation for this phenomenon.

It is well known that the triangular loops in the network tend to inhibit the spreading behaviors, such as the propagation of epidemic [16]. In the context of Naming Game, a network with many triangular loops also has a strong restriction against the growth of word clusters. As mentioned above, clustering structures provide a larger probability for nodes to keep words in their neighborhoods. Consider a case of a cluster of neighboring agents sharing a word. If they are densely connected, it is very hard for other word clusters to invade them; thus, for a network with larger clustering coefficient, the competition among word clusters will take a longer time leading to a slower growing velocity of the average cluster size.

IV. CONCLUSION AND DISCUSSION

In this paper, we have investigated the Naming Game on homogeneous small-world networks and focused on the influence of clustering topology on the dynamical process. A larger clustering coefficient allows nodes to restrict words in their neighborhoods, and thus induces a lower maximum memory in the early stage. We obtain a quantitative relationship between the maximum memory and the clustering coefficient. The clustering also inhibits the growing of word clusters, thus a network with larger clustering coefficient requires a longer crossover time.

It is worth to emphasize that the node's degree of the network also plays an important role in the dynamics

of the Naming Game model [9]. In this paper, we use the homogeneous networks to eliminate the influence induced by varying the node's degree. The relationships between node degree and current dynamics is one of the most interesting issues that deserves further efforts to be studied.

Furthermore, recent works on the issue of Naming Game has revealed that Naming Game model can also describe the spreading of opinions or, from a more general viewpoint, the evolution of communication systems. [17, 18]. Although there exists some different details when applying Naming Game to different fields [6, 19], all of them are needed to capture the essential property of this kind of dynamics, *i.e.*, the system should evolve to a consensus state without any external or global coordination.

Acknowledgments

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